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1999 J. Phys. A: Math. Gen. 32 5907

(http://iopscience.iop.org/0305-4470/32/32/306)

J. Phys. A: Math. Gen. **32** (1999) 5907–5920. Printed in the UK PII: S0305-4470(99)00445-X

Dynamical properties of the classical continuous XY model in a time-dependent magnetic field

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Received 30 December 1998, in final form 7 May 1999

Abstract. The classical continuous XY model—defined as the two-component normalized spin–field—within a time-dependent magnetic field is investigated. It is shown that the dynamics of the spin–field is governed by the elliptic sine–Gordon equation in which the time dependence is built into the time-dependent external magnetic field. This equation is solved by using the covariant Hamilton–Jacobi equation technique and the Bäcklund transformation method. The reasons for the poor dynamics of the model are discussed.

1. Introduction

The classical XY model—as considered in the literature—is usually understood as the model defined on the basis of the classical three-dimensional spin vector $\vec{S} = (S^x, S^y, S^z)$ of unit length. It is assumed that the Hamiltonian contains only two components (S^x, S^y) [1, 2]. Such a model describes quasi-one-dimensional magnets (for example, $CsNiF_3$) [3–5].

In this paper we present a consistent theory of the XY model defined as the classical two-component spin vector $\vec{S} = (S^x, S^y)$ subjected to the normalization condition [6, 7]:

$$(S^x)^2 + (S^y)^2 = 1. \tag{1.1}$$

Interactions in this model are described by the same Hamiltonian as in the previous formulation, i.e.

$$H = -J \sum_{(i,j)} (S_i^x S_j^x + S_i^y S_j^y) \tag{1.2}$$

where $\vec{S}_i$ is the spin vector located in the ith point of the one-dimensional lattice and J is the exchange integral. (The sum runs over the nearest neighbours.)

One should stress that such differences in definitions are not relevant for the equilibrium statistical mechanics description point of view, but they become important in considering topological and dynamical properties of the model. This follows from the fact that topological properties depend on the structure of the field and the dynamics requires definition of an additional mathematical structure. If one considers dynamics on the basis of Hamiltonian formalism one should know the structure of fundamental Poisson brackets [8].

In our approach we define the dynamics of the XY model by the fundamental Poisson brackets in the form:

$$\{S_i^x, S_j^y\} = B\delta_{ij} \qquad \{S_i^x, S_j^x\} = 0 = \{S_i^y, S_j^y\} \tag{1.3}$$

0305-4470/99/325907+14\$30.00

where $B = \text{const}$ and δ_{ij} is the Kronecker symbol.

In order to show important differences in the dynamical behaviour of both approaches we consider the equations of motion. They are the following:

- for the classical three-component spin [9]:

$$\dot{S}^{\alpha} = \varepsilon^{\alpha\beta\gamma}\frac{\partial H}{\partial S^{\beta}}S^{\gamma} \tag{1.4}$$

where $\alpha, \beta, \gamma = 1, 2, 3$ and the repeated indices imply the summation;
- for the classical two-component spin:

$$\dot{S}^{1} = B\frac{\partial H}{\partial S^{2}} \tag{1.5a}$$

$$\dot{S}^{2} = -B\frac{\partial H}{\partial S^{1}} \tag{1.5b}$$

where $S^1 = S^x$ and $S^2 = S^y$.

After differentiating with respect to time the spin-normalization conditions and using (1.4), (1.5) we obtain:
- for the three-component model ($\alpha, \beta, \gamma = 1, 2, 3$):

$$S^{\alpha}\dot{S}^{\alpha} = \varepsilon^{\alpha\beta\gamma}S^{\alpha}\frac{\partial H}{\partial S^{\beta}}S^{\gamma} \tag{1.6}$$

- for the two-component model ($\alpha = 1, 2$):

$$S^{\alpha}\dot{S}^{\alpha} = B\left[S^{1}\frac{\partial H}{\partial S^{2}} - S^{2}\frac{\partial H}{\partial S^{1}}\right]. \tag{1.7}$$

The right-hand side of (1.6) is identically zero whereas the right-hand side of (1.7) is, in general, not zero. This means that equations of motion (1.5) are not consistent with spin-normalization condition (1.1). To avoid such inconsistency one should use Dirac's method in order to obtain correct equations of motion [10]. This method provides one with the generalization of the Hamilton dynamics for the case where constraints inconsistent with the equations of motion exist.

The Dirac method will be described in section 2 for the case of the classical continuous XY model interacting with an external time-dependent magnetic field. In section 3 we will solve the equation obtained in section 2 using the covariant Hamilton–Jacobi equation technique and in section 4 we will compare the results obtained in section 3 with those obtained by using the Bäcklund transformation method.

2. The classical continuous XY model in an external time-dependent magnetic field

The classical continuous XY model—considered in this paper—is defined as a two-component field:

$$\vec{S}(x) = [u(x), v(x)] \tag{2.1}$$

where $x = (t, \vec{x}) \in \mathbb{R}^{d+1}$ and $u = S^1$, $v = S^2$.

It is assumed that the field is subjected to the normalization condition (1.1) which in this case has the form:

$$\forall x \in \mathbb{R}^{d+1} [u(x)]^2 + [v(x)]^2 = 1. \tag{2.2}$$

For the model in an external time-dependent magnetic field the Hamiltonian consists of two parts: the term describing the exchange interaction:

$$H_0 = \frac{K}{2}\int_{\mathbb{R}^d} \mathrm{d}\vec{x}\,[(\vec{\nabla}u)^2 + (\vec{\nabla}v)^2] \tag{2.3a}$$

where K is the exchange integral, and the term describing interaction of the spin–field with the external time-dependent magnetic field $h(t)$ in the direction of the S^1-axis:

$$H_1 = -h(t)\int_{\mathbb{R}^d} \mathrm{d}\vec{x}\,u(x). \tag{2.3b}$$

The fundamental Poisson brackets (1.3) have the form:

$$\{u(\vec{x}), v(\vec{y})\} = B\delta(\vec{x}-\vec{y}) \qquad \{u(\vec{x}), u(\vec{y})\} = 0 = \{v(\vec{x}), v(\vec{y})\}. \tag{2.4}$$

If F and G are arbitrary functionals of u and v, then the Poisson bracket is defined as

$$\{F, G\} = \int_{\mathbb{R}^d} \mathrm{d}\vec{x}\,B\left[\frac{\delta F}{\delta u(\vec{x})}\frac{\delta G}{\delta v(\vec{x})} - \frac{\delta F}{\delta v(\vec{x})}\frac{\delta G}{\delta u(\vec{x})}\right] \tag{2.5}$$

where $\frac{\delta F}{\delta u(\vec{x})}, \frac{\delta F}{\delta v(\vec{x})}, \frac{\delta G}{\delta u(\vec{x})}, \frac{\delta G}{\delta v(\vec{x})}$ are variational derivatives of F and G [8].

Since condition (2.2) is not consistent with Hamilton's equations of motion, obtained in a 'standard' way we have to use Dirac's method [10]. In this method the Hamiltonian of the constrained system is taken in the form:

$$H^* = H + \int_{\mathbb{R}^d} \mathrm{d}\vec{x}\,\lambda(x)\phi(x) \tag{2.6}$$

where H is the Hamiltonian of the unconstrained system and $\phi(x)$ is the constraint given in the form:

$$\phi(x) \approx 0 \tag{2.7}$$

with $\approx$ indicating the so-called Dirac's weak equality. The meaning of this equality is the following: if we are to compute the Poisson brackets of expressions containing the constraint function $\phi(x)$, we cannot set this function to zero as long as we do not evaluate these brackets.

The function $\lambda(x)$ is the Lagrange multiplier determined by Dirac's procedure, which for the considered case will be given below.

Let:

$$H = H_0 + H_1 \tag{2.8}$$

where H_0 and H_1 are given by formulae (2.3) and constraint (2.2) is given as

$$\phi(x) = \tfrac{1}{2}\{[u(x)]^2 + [v(x)]^2 - 1\} \approx 0. \tag{2.9}$$

Equations of motion are obtained on the basis of general formulae:

$$\dot{S}^\alpha = \{S^\alpha, H^*\} \tag{2.10}$$

(where $\alpha = 1, 2$ and $S^1 = u$, $S^2 = v$) and they take the form:

$$\dot{u}(x) = B[-K\Delta v(x) + \lambda(x)v(x)] \tag{2.11a}$$

$$\dot{v}(x) = B[K\Delta u(x) + h(t) - \lambda(x)u(x)] \tag{2.11b}$$

where Δ is the d-dimensional Laplacian (over space coordinates).

In the first step of Dirac's procedure we examine the compatibility of equations (2.11) with primary constraint (2.9). In order to do that let us differentiate (2.9) with respect to time and substitute instead of $\dot{u}$ and $\dot{v}$ the right-hand sides of (2.11). We obtain the following result:

$$-B[K(u\Delta v - v\Delta u) - h(t)v] \approx 0. \tag{2.12}$$

Because the left-hand side of (2.12) does not depend on Lagrange multiplier λ, this equation represents the so-called secondary constraint:

$$\Psi = K(u\Delta v - v\Delta u) - h(t)v \approx 0. \tag{2.12$'$}$$

Constraint (2.12′) ought to be consistent with equations of motion (2.11), therefore we examine the consistency condition again. Repeating the above-described procedure we obtain

$$\begin{aligned} &BK\Delta\lambda - Bhu\lambda - BK\{K[u\Delta(\Delta u) + v\Delta(\Delta v) - (\Delta u)^2 - (\Delta v)^2] - 2h\Delta u\} \\ &\qquad + Bh^2 + \frac{\mathrm{d}h}{\mathrm{d}t}v \approx 0 \end{aligned} \tag{2.13}$$

which is the equation for λ.

Altogether the system of equations describing the dynamics of the classical continuous XY model in an external magnetic field consists of equations of motion (2.11), primary constraint (2.9), secondary constraint (2.12′) and equation for the Lagrange multiplier (2.13).

In order to simplify that system it is convenient to introduce a new variable $\varphi(x)$ such that:

$$\begin{aligned} u(x) &= \cos\varphi(x) \\ v(x) &= \sin\varphi(x). \end{aligned} \tag{2.14}$$

This function represents the angle between the S^1-axis and the spin orientation at point $\vec{x}$. Therefore it should be real.

The simplified system of equations is now:

- equation of motion:

$$\dot{\varphi} = B[-K(\vec{\nabla}\varphi)^2 + h\cos\varphi - \lambda] \tag{2.15a}$$

- secondary constraint:

$$K\Delta\varphi = h(t)\sin\varphi \tag{2.15b}$$

and the equation to fix $\lambda(x)$:

$$B(K\Delta - h\cos\varphi)[\lambda - h\cos\varphi + K(\vec{\nabla}\varphi)^2] + \frac{\mathrm{d}h}{\mathrm{d}t}\sin\varphi = 0. \tag{2.15c}$$

(Primary constraint (2.9) is satisfied automatically.)

The solution of our problem consists in finding such functions $\varphi(x)$ and $\lambda(x)$ so that equations (2.15) are satisfied simultaneously.

Instead of solving equation (2.15*c*) with respect to λ and inserting the obtained solution into (2.15*a*) in order to get the equation for φ, we eliminate λ from those equations. In order to do that let us define the operator:

$$\hat{D} = K\Delta - h\cos\varphi. \tag{2.16}$$

Operating on both sides of (2.15*a*) with (2.16) and using (2.15*c*) we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}[K\Delta\varphi - h\sin\varphi] = 0. \tag{2.17}$$

Thus equations (2.15*a*) and (2.15*c*) reduce to (2.17) which states that the expression in the square bracket is the first integral of them. This requirement establishes a restriction for admissible initial conditions. The set of these conditions forms a submanifold of the starting phase space of the model. Furthermore, the consistency with equation (2.15*b*) requires that this expression is to be equal to zero. Therefore, we are to choose those initial conditions so that such a consistency occurs.

The above analysis has shown that the only equation describing the dynamics of the classical continuous XY model in an external time-dependent magnetic field is the elliptic sine–Gordon equation (2.15*b*). This equation corresponds formally to that obtained by Mikeska [1], however, there is an important difference between them: our equation does not contain the time derivative explicitly and the time dependence of φ is governed by the time-dependent external magnetic field. This is the only way to start the dynamics of the model. One can say that the dynamics of this model is 'exotic' because it depends on time through the 'external' function $h(t)$.

3. The covariant Hamilton–Jacobi equation technique for the elliptic sine–Gordon equation

It was shown in section 2 that the only equation describing the dynamics of the classical continuous XY model in a time-dependent magnetic field is the elliptic sine–Gordon equation (2.15*b*). This equation did not contain the time derivative, but the time dependence was built into the magnetic field. This is the characteristic feature of the model defined as a two-component spin–field with constraint (2.2).

In this section we solve this equation by using the covariant Hamilton–Jacobi equation technique. This technique reduces the problem of solving the differential equation of the second order to solving the differential equation of the first order [11, 12].

We rewrite equation (2.15*b*) in the form:

$$\Delta\varphi = q(t)\sin\varphi \tag{3.1}$$

where $q(t) = \frac{h(t)}{K}$.

This equation can be expressed in the form of the covariant Euler–Lagrange equation:

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right] - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \tag{3.2}$$

where $\mathcal{L}$ is the Lagrangian density given by

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - 2q(t)\cos^2\tfrac{\varphi}{2}. \tag{3.3}$$

Defining the generalized impulse:

$$P^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \tag{3.4}$$

and the Hamiltonian density:

$$\mathcal{H} = P^\mu(\partial_\mu\varphi) - \mathcal{L} \tag{3.5}$$

the equations of motion are the following:

$$\partial_\mu\varphi = \frac{\partial\mathcal{H}}{\partial P^\mu} \tag{3.6a}$$

$$\partial_\mu P^\mu = -\frac{\partial\mathcal{H}}{\partial\varphi}. \tag{3.6b}$$

The corresponding covariant Hamilton–Jacobi equation is [11, 12]:

$$\mathcal{H}\left(\varphi, \frac{\partial S^\mu}{\partial\varphi}, x\right) + \partial_\mu S^\mu = 0 \tag{3.7}$$

where S^μ is defined as

$$\frac{\partial S^\mu}{\partial\varphi} = P^\mu. \tag{3.8}$$

In the present case:

$$\mathcal{H} = \tfrac{1}{2} P^{\mu} P_{\mu} + 2q \cos^2 \tfrac{\varphi}{2} \tag{3.9}$$

and so the covariant Hamilton–Jacobi equation takes the explicit form:

$$\frac{1}{2} \frac{\partial S^{\mu}}{\partial \varphi} \frac{\partial S_{\mu}}{\partial \varphi} + 2q \cos^2 \frac{\varphi}{2} + \partial_{\mu} S^{\mu} = 0. \tag{3.10}$$

Because the Hamiltonian density does not depend on x explicitly, equation (3.10) can therefore be separated into the form of two independent equations:

$$\frac{1}{2} \frac{\partial S^{\mu}}{\partial \varphi} \frac{\partial S_{\mu}}{\partial \varphi} + 2q \cos^2 \frac{\varphi}{2} = E \tag{3.11a}$$

$$\partial_{\mu} S^{\mu} + E = 0 \tag{3.11b}$$

where E is the separation constant.

Inserting $\frac{\partial S^{\mu}}{\partial \varphi} = P^{\mu} = \partial^{\mu} \varphi$ into (3.11a) one obtains:

$$\tfrac{1}{2} (\partial^{\mu} \varphi)(\partial_{\mu} \varphi) + 2q \cos^2 \tfrac{\varphi}{2} = E. \tag{3.12}$$

Let us consider the meaning of the separation constant E. As $\varphi(x)$ is a real function $(\partial_{\mu}\varphi)(\partial^{\mu}\varphi) \geqslant 0$. Therefore, following (3.12) we have:

$$E - 2q \cos^2 \tfrac{\varphi}{2} \geqslant 0 \qquad \forall \varphi \in [0, 2\pi]. \tag{3.13}$$

In particular, equation (3.13) should be satisfied for the maximal value of $\cos^2 \frac{\varphi}{2}$, thus:

$$E - 2q \geqslant 0. \tag{3.14}$$

The exact value of E can be obtained from (3.12) by taking an appropriate boundary condition for φ and its gradient $\partial_{\mu}\varphi$. Moreover, it results from (3.12) that this value depends on $q(t)$.

With the aid of the new variable chosen in the form [13]:

$$V = \int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\xi}{\sqrt{2E - 4q \cos^2 \frac{\xi}{2}}} \tag{3.15}$$

where $\varphi_0 = \text{const}$, equation (3.12) can be rewritten as

$$(\partial_{\mu} V)(\partial^{\mu} V) = 1 \tag{3.16}$$

which is the eikonal equation. (Following (3.13) V is a real function.)

The procedure of obtaining solutions of the Hamilton–Jacobi equation (3.12) consists in computing integral (3.15) and solving the obtained equation with respect to φ.

Exact solutions of (3.16) obtained in the way of group-theoretic analysis are given in [14]. In a further analysis we take the simplest solution in the form:

$$V(x) = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} x^i \tag{3.17}$$

where d is the dimension.

The quantity q in (3.15) is a function of time which, in general, may be positive, negative or equal to zero. Therefore, equation (3.15) can be written as

$$V = \int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\xi}{\sqrt{2E \pm 4|q| \cos^2 \frac{\xi}{2}}} \tag{3.18a}$$

where the sign '+' is for $q < 0$ and '−' is for $q > 0$ and

$$V = \frac{\varphi - \varphi_0}{\sqrt{2E}} \tag{3.18b}$$

for $q = 0$.

Integrals (3.18a) can be transformed into Legendre's form of the elliptic integral of the first kind [15]. They are the following (for simplicity we assume that $\varphi_0 = 0$):

(1) For $q < 0$:

$$V = \frac{k}{\sqrt{|q|}} F\left(\frac{\varphi}{2}, k\right) \tag{3.19}$$

where F is the Legendre elliptic integral of the first kind and $k = \sqrt{\frac{2|q|}{E+2|q|}}$ is the modulus.

(2) For $q > 0$:

$$V = \frac{k}{\sqrt{|q|}} \left[K(k) - F\left(\frac{\pi - \varphi}{2}, k\right)\right] \tag{3.20}$$

where $k = \sqrt{\frac{2|q|}{E}}$ and $K(k) = F(\frac{\pi}{2}, k)$ is the complete elliptic integral.

Inverting (3.19) and (3.20) with respect to φ we obtain solutions given by formulae:

(1)

$$\varphi = 2am\left(\frac{\sqrt{|q|}}{k} V, k\right) \tag{3.21}$$

where $am(u, k)$ is the Jacobian elliptic function called the amplitude of u;

(2)

$$\varphi = \pi + 2am\left(\frac{\sqrt{|q|}}{k} V - K, k\right). \tag{3.22$'$}$$

Following the properties of the function $am(u, k)$ equation (3.22′) takes the form:

$$\varphi = 2 \arctan\left[k' tn\left(\frac{\sqrt{|q|}}{k} V, k\right)\right] \tag{3.22}$$

where $k' = \sqrt{1 - k^2}$ and $tn(u, k) = \tan(am(u, k))$ [15].

Solutions (3.21) and (3.22) have such a property that in the limit $|q| \to 0$ they tend to (3.18*b*) with $\varphi_0 = 0$.

Figures 1(*a*) and 1(*b*) present two-dimensional plots of solutions (3.21) and (3.22) in the plane $y = 0$, respectively. The values of the magnetic field are represented by $k = 0.999\,999$ in both cases.

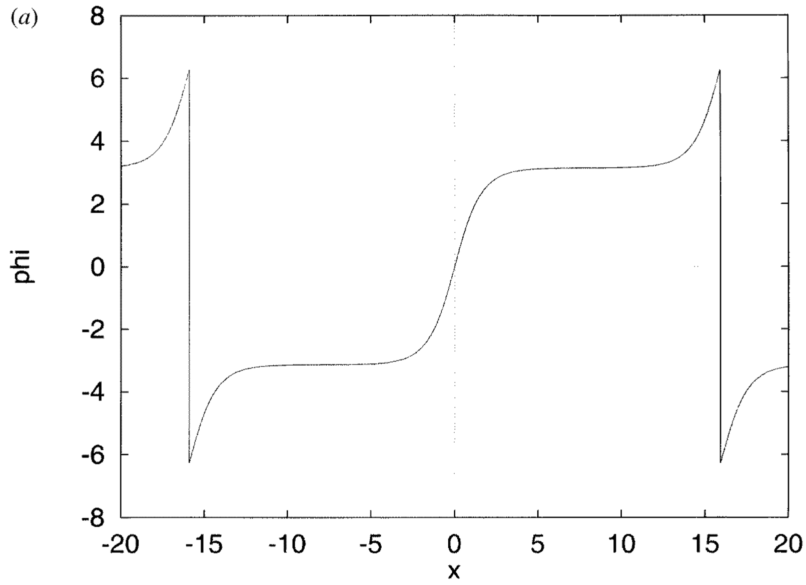
In the first case $q < 0$ (figure 1(*a*)). For simplicity we assume that $q = 2k^2$. Therefore the separation constant is $E = 4(1 - k^2)$.

The spin phase exhibits a rapid change of values in the centre of the domain and near their boundaries. Behaviour in the centre has a soliton-like character. In this region the competition appears between the exchange effect (important in short distances) and interaction with an external magnetic field. For large distances the influence of the magnetic field has predominant significance.

The solution for $q > 0$ (figure 1(*b*)) in which $E = 4$ (for $q = 2k^2$) is a simple modification of the previous case (which is seen in (3.22′)). It exhibits the 'boundary effect', but in the centre of the domain the spins are parallel.

4. The Bäcklund transformation method for the elliptic sine–Gordon equation

The main qualitative result of section 3 is the following: the solutions of the problem depend on time through the time-dependent external magnetic field (appearing as a coefficient in the elliptic sine–Gordon equation). It turned out that the sign of the function $h(t)$ had an important significance. Dependent on that sign we had three branches of solutions given by (3.21), (3.18*b*) and (3.22). But in the limit $|q| \to 0$ remained the only solution given by (3.18*b*).



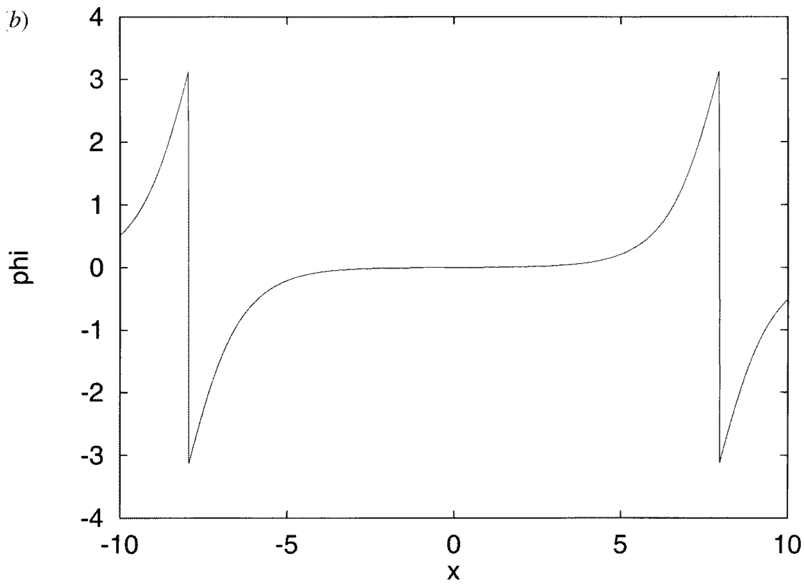


Figure 1. (*a*) The phase $\varphi(x, y)$ of the XY spin–field configuration in the time-dependent magnetic field $h < 0$ given by formula (3.21) in the plane $y = 0$ for $k = 0.999\,999$, $q = 2k^2$ and $E = 4(1 - k^2)$. (*b*) The phase $\varphi(x, y)$ of the XY spin–field configuration in the time-dependent magnetic field $h > 0$ given by formula (3.22) in the plane $y = 0$ for $k = 0.999\,999$, $q = 2k^2$ and $E = 4$.

In this section we verify the above results on the basis of the Bäcklund transformation method for the equation in two spatial dimensions.

The main idea of this method is to find such relations between two solutions of the considered equation given in the form of differential equations so that it is possible to integrate them in a simpler way. (A precise definition of the Bäcklund transformation is given in [16, 17].)

In the considered case equation (3.1) has the form:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi = q(t)\sin\varphi \tag{4.1}$$

which can be written as:

$$\partial_z\partial_{\bar z}\varphi = g(t)\sin\varphi \tag{4.2}$$

where φ is a real function, $z = x + \mathrm{i}y$, $\bar z = x - \mathrm{i}y$, $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y})$, $\partial_{\bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$, $g(t) = \frac{1}{4}q(t)$.

The Bäcklund transformation for equation (4.2) is the following:

$$\partial_z\Psi_1 = \xi_1(\theta)\sin\Psi_2 \tag{4.3a}$$

$$\partial_{\bar z}\Psi_2 = \xi_2(\theta)\sin\Psi_1 \tag{4.3b}$$

where

$$\Psi_1 = \tfrac{1}{2}(\varphi_1 + \beta\varphi_2) \tag{4.4a}$$

$$\Psi_2 = \tfrac{1}{2}(\varphi_1 - \beta\varphi_2) \tag{4.4b}$$

and θ is called the spectral or Bäcklund transformation parameter.

The functions φ_1 and $\beta\varphi_2$ are two solutions of equation (4.2), β is constant and

$$\xi_1\xi_2 = g. \tag{4.5}$$

Before obtaining solutions of (4.2) let us consider the complex conjugation of functions Ψ_1 and Ψ_2. It is easy to see that the following symmetry relations hold (bars denote the complex conjugation):

$$\bar\Psi_1 = \Psi_1 \qquad \bar\Psi_2 = \Psi_2 \tag{4.6}$$

when β is a real number;

$$\bar\Psi_1 = \Psi_2 \qquad \bar\Psi_2 = \Psi_1 \tag{4.7}$$

when β is a pure imaginary number.

At first let us consider the second case given by (4.7) [18]. Taking the complex conjugation of both sides of (4.3a) and comparing with (4.3b) we obtain that

$$\xi_2 = \bar\xi_1. \tag{4.8}$$

Substituting (4.8) into (4.5) we have

$$|\xi_1|^2 = g. \tag{4.9}$$

Thus $\xi_1 = \sqrt{g}\mathrm{e}^{\mathrm{i}\theta}$, where θ is the spectral parameter.

It results from (4.9) that in this case the Bäcklund transformation exists when g is positive.

In the first case (equation (4.6)) let us assume for simplicity that $\beta = 1$.

One can rewrite equations (4.3) in the form:

$$\partial_z\left(\frac{\varphi_1+\varphi_2}{2}\right) = \xi_1\sin\left(\frac{\varphi_1-\varphi_2}{2}\right) \tag{4.10a}$$

$$\partial_{\bar z}\left(\frac{\varphi_1-\varphi_2}{2}\right) = \xi_2\sin\left(\frac{\varphi_1+\varphi_2}{2}\right). \tag{4.10b}$$

Because $\varphi_2 = 0$ is a solution of (4.2) one can thus substitute it into (4.10).

We obtain the following equations for φ_1:

$$\partial_z\left(\frac{\varphi_1}{2}\right) = \xi_1\sin\left(\frac{\varphi_1}{2}\right) \tag{4.11a}$$

$$\partial_{\bar z}\left(\frac{\varphi_1}{2}\right) = \xi_2\sin\left(\frac{\varphi_1}{2}\right). \tag{4.11b}$$

Taking complex conjugation of (4.11*b*) and comparing the obtained equation with (4.11*a*) we obtain formula (4.8) again. This means that equation (4.2) has real solutions when the parameter g is positive.

The solution of equations (4.11) has he form:

$$\varphi_1 = 4 \arctan\{c \exp[\xi_1(\theta)z + \bar{\xi}_1(\theta)\bar{z}]\} \tag{4.12}$$

or

$$\varphi_1 = 4 \arctan\{c \exp[2\sqrt{g}(x \cos\theta - y \sin\theta)]\} \tag{4.12$'$}$$

where c is a real constant, which is the one-soliton-like solution for great values of g. (The method of obtaining multiple-soliton-like solutions is described in [18].)

The above considerations concerned the situation when the sign of the function $g(t)$ in equation (4.2) was positive. Moreover, it can be equal to zero or negative.

In the case where $g = 0$, equation (4.2) takes the form:

$$\partial_z \partial_{\bar{z}} \varphi = 0. \tag{4.13}$$

The general solution of (4.13) in the form of a real function is

$$\varphi = f(z) + \overline{f(z)} \tag{4.14}$$

where f is an arbitrary function of the argument z and $\bar{f}$ is its complex conjugation. In the case where $g < 0$, equation (4.2) can be written as

$$\partial_z \partial_{\bar{z}} \varphi = -|g| \sin\varphi. \tag{4.15}$$

Using the Bäcklund transformation method one can obtain solutions of (4.15) in the form of complex functions, which here have no physical interpretation. However, the real solution of equation (4.15) obtained using the property of the trigonometric function sine can be written in the form

$$\varphi = \pi + \varphi_1 \tag{4.16}$$

where φ_1 is the solution of equation (4.2) with $g > 0$.

Solution (4.16) can be interpreted in this way so that the change of direction of the magnetic field gives the spin–field configuration turned by π.

Let us consider the properties of solutions (4.12) and (4.16) in the case where $|g| \to 0$. Following (4.9) and (4.8) $\xi_i \to 0$, $i = 1, 2$, in that limit. Thus solution (4.12) takes the form:

$$\varphi_1 = 4 \arctan c \tag{4.17}$$

whereas

$$\varphi = \pi + 4 \arctan c_0. \tag{4.18}$$

In order to obtain consistency of the above solutions we compare (4.18) with (4.17). The consistency condition is

$$c_0 = \frac{c - 1}{c + 1} \tag{4.19}$$

where $c \neq -1$.

Solution (4.14), for $g = 0$, ought to be consistent with (4.17) and (4.18). This consistency condition reads $\varphi = 4 \arctan c$.

Thus the solutions obtained using Bäcklund transformation have similar properties as those obtained on the basis of the Hamilton–Jacobi equation, but they differ in their functional forms.

However, it is possible to compare these solutions. In order to do this let us assume that $k = 1$ in equation (3.19).

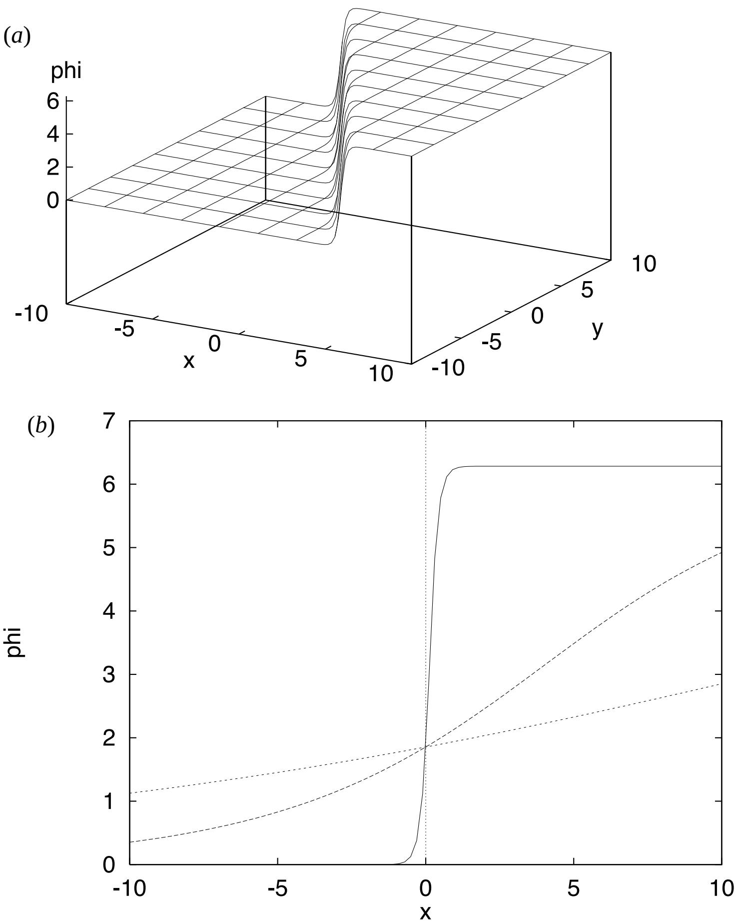


Figure 2. (*a*) The phase $\varphi(x, y)$ of the XY spin–field configuration in the time-dependent magnetic field $h > 0$ given by formula (4.12′) for $c = 0.5$, $\theta = \frac{11\pi}{6}$, $g = 10$. (*b*) Sections of configuration given in (*a*) by the plane $y = 0$ for $g = 10$ (——), $g = 0.01$ (– – –), $g = 0.001$ (· · · · · ·).

Following the formula [15]:

$$F\left(\frac{\varphi}{2}, 1\right) = \ln\tan\left(\frac{\pi + \varphi}{4}\right) \tag{4.20}$$

we have

$$\varphi = 4\arctan\left\{\exp\left[2\sqrt{|g|}\frac{(x+y)}{\sqrt{2}}\right]\right\} - \pi \tag{4.21}$$

which corresponds to (4.16).

Figure 2(*a*) presents a three-dimensional plot of the solution (4.12′). The values of the parameters are the following: $c = 0.5$, $\theta = \frac{11\pi}{6}$ and $g = 10$. It exhibits a localization characteristic of a one-soliton-like solution. The behaviour of this solution in a decreasing magnetic field is shown in figure 2(*b*). The plots were made in the plane $y = 0$. The solutions become increasingly flat when the field decreases and tend to a constant value $\varphi = 4\arctan c$ when the field disappears. Thus, the soliton decays in the decreasing field. Applying the increasing magnetic field of opposite direction we can create the soliton (with the phase turned by π) again. The results for this case are presented in figures 3(*a*) and 3(*b*). The functional

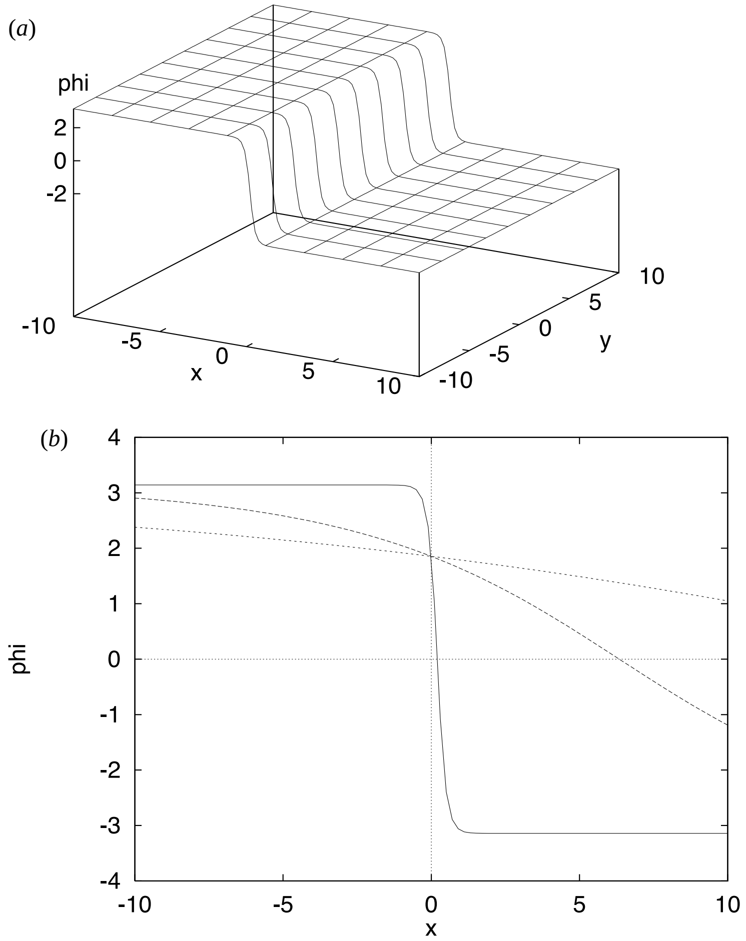


Figure 3. (*a*) The phase $\varphi(x, y)$ of the XY spin–field configuration in the time-dependent magnetic field $h < 0$ given by formula (4.16) for $c_0 = -\frac{1}{3}$, $\theta = \frac{11\pi}{6}$, $g = 10$. (*b*) Sections of configurations given in (*a*) by the plane $y = 0$ for $g = 10$ (——), $g = 0.01$ (– – –), $g = 0.001$ (· · · · · ·).

form of the solution is given by formula (4.16). The constant coefficient c_0 is given by (4.19) and is equal to: $-\frac{1}{3}$. The parameters θ and g have the same values as in figures 2(*a*) and 2(*b*).

5. Conclusions

In this paper we have analysed the properties of the classical continuous XY model in a time-dependent external magnetic field.

The model was defined as a two-component field on $\mathbb{R}^d$ with the normalization condition given by equation (2.2). It appeared that the dynamics of such a defined model had to be described on the basis of Dirac's method for constrained systems.

The analyses showed that the only equation describing the evolution of the model was the elliptic sine–Gordon equation with the time-dependent coefficient (equation (3.1)). This equation was simultaneously the first integral of equations of motion (2.15*a*) and (2.15*c*), as well as an additional constraint imposed on the starting space defined by normalization condition (2.2). This starting space has the structure of the fibre bundle with the total space: $M = \bigcup_{x\in\mathbb{R}^d} (S^1)_x$, where $(S^1)_x$ is a circle at point $x \in \mathbb{R}^d$ and the base space $\mathbb{R}^d$, whereas the resulting space is given by solutions of equation (2.15*b*). The specific feature of this equation is the lack of time derivative in it. Thus, the dynamics of the spin–field is governed by the time dependence of the external magnetic field.

Such a 'strange' behaviour of the model results from the fact that normalization condition (2.2) is not invariant of the Lie algebra defined by (2.4).

Equation (3.1) was solved in two ways:

(i) using the covariant Hamilton–Jacobi equation;
(ii) on the basis of the Bäcklund transformation method.

In the first case we considered the problem in the space $\mathbb{R}^d$ of arbitrary dimension d, whereas in the second case we considered the problem in the space $\mathbb{R}^2$.

The results are qualitatively similar: dependent on the sign of the function $h(t)$ we have obtained three branches of solutions: for $h(t) < 0$, $h(t) = 0$ and $h(t) > 0$. When $|h| \to 0$ the solutions obtained on the basis of both methods transform continuously to a 'static' solution (i.e. for $h = 0$).

The solutions obtained on the basis of the Hamilton–Jacobi technique have the form of Jacobian elliptic functions and exhibit a domain structure, whereas the solutions obtained on the basis of the Bäcklund transformation method have the form of one-soliton-like solutions for great values of $h(t)$ and they decay when the field disappears. Moreover, they are the special case of those obtained on the basis of the Hamilton–Jacobi method. However, the Bäcklund transformation method provides the possibility of generating multisoliton solutions.

One can show that our approach to the classical XY model leads to the same structure of equations of motion as the approaches considered in [2] for the classical XY model and in [9] for the classical Heisenberg model with the constraint: $S^z - B \approx 0$, where B is the constant defined in equation (1.3).

The elliptic sine–Gordon equation has been obtained in this paper as the equation of motion of the classical continuous XY model in an external magnetic field. The important feature of this equation is the fact that the coefficient appearing by the sine function on the right-hand side of it is a function of time and may change the sign, dependent on the direction of the magnetic field.

The equation with $q = \text{const} > 0$ was investigated by many authors. Generally, this equation describes static (i.e. time-independent) nonlinear phenomena in two-dimensional systems of condensed matter physics and field theory [19]. In particular, the elliptic sine–Gordon equation can be used to describe topological defects in magnetic structures [20–24] and its solutions may be used in describing the propagation of magnetic flux through a large two-dimensional Josephson tunnelling junction [18]. The authors used the Bäcklund transformation method [18, 20], the direct Hirota method [22, 23], the inverse scattering method [24–28] and the ansatz: $\varphi = 4\arctan[X(x)Y(y)]$ [21].

The results obtained in this paper on the basis of the Bäcklund transformation method (for $q = 1$) are analogical to those obtained in [18], however the results presented in [20] (obtained by the same method) are more general and describe topological defects in incommensurate magnetic and crystal structures. Similarly, the results obtained by the inverse scattering method are more general and provide the possibility to describe vortices and vortex dipoles [27, 28].

All these methods were used only in the case where $q > 0$, whereas the Hamilton–Jacobi technique also provides the possibility of solutions for $q < 0$. Thus the elliptic sine–Gordon equation with the negative right-hand side needs more detailed investigation using other methods.

Acknowledgments

We would like to thank Professor K Rościszewski for discussions, critical reading of the manuscript and valuable suggestions concerning the form of this paper. We are also grateful to Dr M Fiałkowski and Dr A Kapanowski for discussions.

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